Home exam: course by Andrés Navas

Problem 1. *a*) Find a (non-minimal) map $T : X \to X$ and a function $\varphi : X \to \mathbb{R}$ with bounded Birkhoff sums,

$$\forall x \in X, n \in \mathbb{N} \quad |B_n(\varphi)| \leq C,$$

for which the corresponding skew product admits no invariant continuous section.

b) The same question, but now T has to be topologically transitive.

Problem 2. Prove the Hedlund Theorem for group actions: if G acts on X minimally, and $c : G \times X \to \mathbb{R}$ is an additive cocycle, then the following conditions are equivalent:

i) $\exists x_0 \in X, \exists C > 0$ s.t. $|c(g, x_0)| \leq C \; \forall g \in G$

ii) $\forall x_0 \in X, \exists C > 0$ s.t. $|c(g, x_0)| \leq C \; \forall g \in G$

iii) The cohomological equation $\sigma(gx) - \sigma(x) = c(g, x)$ has a continuous solution.

Problem 3. Let $G = \mathbb{Z}^d$ be a group, for an action of which we are given a bounded cocycle $c(\cdot, \cdot)$. Prove that then a sequence of almost invariant sections could be written explicitly as

$$\sigma_n(x) = \frac{1}{n^d} \sum_{0 \leq i_k < n} c(T_1^{i_1} \dots T_d^{i_d}, x)$$

Problem 4. Prove that the following are equivalent:

i) $\int_{S^1} \log Dg(x) d\mu(x) = 0$ for any g-invariant measure μ;

ii) g has no hyperbolic periodic fixed points

Problem 5. A center of a bounded set is the center of a ball of the smallest radius containing this set.

a) Find the center of a triangle;

b) Prove that the center belongs to convex closure of a set.

Problem 6. *a*) Prove that for an irrational rotation of the circle for any continuous function $\varphi \in C(S^1)$ one has a (uniform in $x \in S^1$) convergence

$$\frac{S_n \varphi(x)}{n} \to \int_{S^1} \varphi(x) \, dx.$$

b) Prove that the cardinality of the set of $\{i : 0 \leq i < n \mid \theta + i\alpha \in I\}$, divided by n, tends to the length of an interval I.

Problem 7 (Denjoy-Koksma inequality)**.** For a function φ on a circle with bounded variation show that

$$\forall \theta, n \quad |S_{q_n}\varphi(\theta) - q_n \int_{S^1} \varphi(x)dx| \leq \operatorname{Var}\varphi,$$

where $\frac{p_n}{q_n}$ is a sequence of good approximations of α, that is, incomplete fractions corresponding to the decomposition of α into a continuous fraction.

Problem 8. Let G be a semigroup acting by isometries on a $CAT(0)$-space. Assume that there is a point with a bounded orbit. Then, the action has a fixed point.

Problem 9. Prove the existence of a barycenter in the Cartan sense for the functional given by $w \mapsto \int_H d^2(w,z)d\mu(z)$

Problem 10. Prove that the procedure of constructing a barycenter by induction converges to a point and prove the inequality

$$d(\operatorname{bar}_n(w_1, \ldots w_n), \operatorname{bar}_n(w'_1, \ldots w'_n)) \leq \frac{1}{n}\sum_1^n d(w_i, w'_i)$$

Problem 11. Prove the existence and uniqueness of a center for uniformly convex Banach spaces.

Problem 12 (Furstenberg example)**.** Let $f : T^2 \to T^2$ be a map defined as $(x,y) \mapsto (x+\alpha, y+\varphi(x))$, where α is irrational, and $\int_{S^1}\varphi(x)dx = 0$. Prove that

a) The map f is minimal if and only if the cohomological equation $\varphi(x) = h(x+\alpha) - h(x)$ has no continuous solution;

b) Show that there exists α and φ such that the corresponding cohomological equation has a measurable solution, but no continuous one;

c) Construct such a map f that is minimal, but not ergodic.

Problem 13. Let G be a group of area-preserving diffeomorphisms of a compact orientable surface S.

a) Associate to it a skew product action on the total space $S \times \mathbb{H}$, where $\mathbb{H}$ is a hyperbolic plane, such that the fiberwise actions are isometries.

b) Translate to this framework the isometries version of Hedlund theorem: what is the resulting statement?